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Discrete Mathematics 308 (2008) 303–307
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Resolvable 4-cycle group divisible designs with two associate classes: Part size even

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Received 15 December 2004; received in revised form 10 October 2005; accepted 27 November 2006
Available online 2 June 2007

Abstract

Let $\lambda_1 K_a$ denote the graph on a vertices with λ_1 edges between every pair of vertices. Take p copies of this graph $\lambda_1 K_a$, and join each pair of vertices in different copies with λ_2 edges. The resulting graph is denoted by $K(a, p; \lambda_1, \lambda_2)$, a graph that was of particular interest to Bose and Shimamoto in their study of group divisible designs with two associate classes. The existence of z-cycle decompositions of this graph have been found when $z \in \{3, 4\}$. In this paper we consider resolvable decompositions, finding necessary and sufficient conditions for a 4-cycle factorization of $K(a, p; \lambda_1, \lambda_2)$ (when λ_1 is even) or of $K(a, p; \lambda_1, \lambda_2)$ minus a 1-factor (when λ_1 is odd) whenever a is even.

Keywords: Resolvable; Cycle systems; Associate classes; 2-factorizations

1. Introduction

In this paper, graphs usually contain multiple edges. In particular, if G is a simple graph, then let λG denote the multigraph formed by replacing each edge in G with λ edges. Let C_z denote a cycle of length z.

For any two vertex-disjoint graphs G and H, define $G \vee_\lambda H$ to be the graph formed from the union of G and H by joining each vertex in G to each vertex in H with exactly λ edges; if $\lambda = 1$ then this may be represented by simply $G \vee H$. Let $K(a, p; \lambda_1, \lambda_2)$ denote the graph formed from p vertex-disjoint copies of the multigraph $\lambda_1 K_a$ by joining each pair of vertices in different copies of K_a with λ_2 edges. It will be useful to refer to edges in $K(a, p; \lambda_1, \lambda_2)$ that join vertices in different copies of $\lambda_1 K_a$ as being *mixed* edges, and edges joining vertices in the same copy of $\lambda_1 K_a$ as being *pure* edges. The vertex sets of the p copies of $\lambda_1 K_a$ will usually be $V \times \{i\}$ with $0 \leqslant i < p$ for some set V, where $|V| = a$. Then for each pair $\{u, v\} \subseteq V$, with $0 \leqslant i < j < p$, it will be useful to refer to the edges $\{(u, i), (v, j)\}$ and $\{(u, j), (v, i)\}$ as being the *corresponding mixed edges* to the edge $\{u, v\}$ in K_a. Also, for each $u \in V$ we will refer to the edges $\{(u, i), (u, j)\}$ as being the *horizontal* mixed edges.

An H-decomposition of a graph G is an ordered pair (V, C), where V is the vertex set of G and C is a set of copies of H such that each edge in G occurs in exactly one graph in C. When the actual vertex set V is of no interest, it will cause no confusion to refer to the decomposition by simply C. There has been considerable interest over the past 20 years in H-decompositions of various graphs, such as complete graphs and complete multipartite graphs, especially when H is a

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doi:10.1016/j.disc.2006.11.043

cycle (see [1,3,8,10,11], for example). More recently, the existence problem for C_z-decompositions of $K(a, p; \lambda_1, \lambda_2)$ for $z = 3$ [4,6] and for $z = 4$ [5] has been solved. Such decompositions are known as C_z *group divisible designs with two associate classes*, following the notation of Bose and Shimamoto who considered the existence problem for K_z group divisible designs [2]. (The reason for this name is that the structure can be thought of as partitioning *ap* symbols (i.e. vertices) into p sets of size a in such a way that symbols that are in the same set in the partition occur together in λ_1 blocks, and are known as *first associates*, whereas symbols that are in different sets in the partition occur together in λ_2 blocks, and are known as *second associates*.)

In an H-decomposition (V, C) of a graph G, a *parallel class* is a subset S of C such that each vertex in V occurs in exactly one copy of H in S. The decomposition (V, C) is said to be *resolvable* if C can be partitioned into parallel classes. In this paper we completely settle the existence problem for resolvable C_4-decompositions of $K(a, p; \lambda_1, \lambda_2)$, or of $K(a, p; \lambda_1, \lambda_2)$ minus a 1-factor, when a is even. A resolvable C_4-decomposition is also known as a C_4-*factorization*, and a parallel class in a resolvable C_4-decomposition is also known as a C_4-*factor*.

Let $G[V]$ denote the subgraph of G induced by the vertex set V.

2. Some preliminary results

A *near* C_4-factor of G is a spanning subgraph of G in which one component is K_2 and all others are C_4. A partition of $E(G)$ in which each element induces a near C_4-factor is called a *near* C_4-factorization of G. We can easily obtain the following result, which is of some interest in its own right.

Theorem 1. *For all $n \geqslant 1$ there exists a near C_4-factorization of K_{4n+2}.*

Proof. Let $V(K_{4n+2}) = \mathbb{Z}_{2n+1} \times \mathbb{Z}_2$. Then

$$B = \bigcup_{i=0}^{2n} \{\{(i, 0), (i, 1)\}, ((j + i, 0), (-j + i, 1), (j + i, 1), (-j + i, 0)) | 1 \leqslant j \leqslant n\}$$

provides the required resolvable decomposition. $\square$

In any near C_4-factorization S of K_{4n+2}, it is clear that the set $F(S)$ of copies of K_2 forms a 1-factor of K_{4n+2}. Let $\{F_0, F_1, \ldots, F_{4n}\}$ be a 1-factorization of K_{4n+2}. By simply renaming vertices in S, it is clear that for $0 \leqslant i \leqslant 4n$ we can form a near C_4-factorization S_i of K_{4n+2} in which $F(S_i) = F_i$.

This observation is especially useful here for the following reason. Let $a = 4n + 2$. For $0 \leqslant i \leqslant 4n$ and for $j = 0, 1$, let $\mathscr{F}_j(S_i)$ be the set of 4-cycles formed by deleting F_i from S_i and renaming each vertex u in each 4-cycle with the vertex (u, j). Then

$$\mathscr{F}(S_i) = \mathscr{F}_0(S_i) \cup \mathscr{F}_1(S_i) \cup \{((x, 0), (y, 1), (x, 1), (y, 0)) | \{x, y\} \in E(F_i)\}$$

is a C_4-factorization of the graph formed by joining two copies of K_a with the 1-factor formed by the mixed edges corresponding to the edges in F_i. Also note then that $\bigcup_{i=0}^{4n} \mathscr{F}(S_i)$ is a C_4-factorization of $(4n + 1)K_a \vee (4n + 1) K_a - \{\{(u, 0), (u, 1)\} \mid u \in V\}$, where $V = V(K_a)$. So we can easily get the following result.

Lemma 1. *Let $a = 4n + 2$. There exists a C_4-factorization of $aK_a \vee aK_a$.*

Proof. Extending the notation developed in the previous paragraph, define

$$\mathscr{F}'(S_i) = \mathscr{F}_0(S_i) \cup \mathscr{F}_1(S_i) \cup \{((x, 0), (x, 1), (y, 1), (y, 0)) \mid \{x, y\} \in E(F_i)\}.$$

Then $(\bigcup_{i=0}^{4n} \mathscr{F}(S_i)) \cup \mathscr{F}'(S_0)$ is a C_4-factorization of $aK_a \vee aK_a$. $\square$

We can also obtain Lemma 3 below with a much more sophisticated use of this observation together with the following extremely useful result, essentially proved by Stern and Lenz (see [12] and [9] p.158]).

Lemma 2. *Let G be a regular graph. Suppose there exists a partition of the vertex set of G into two sets of equal size, V_1 and V_2 such that*

- $G_1 = G[V_1]$ *and* $G_2 = G[V_2]$ *are isomorphic simple regular graphs, and*
- *there exists an isomorphism f from G_1 to G_2 such that $\{v, f(v)\}$ is an edge in G for each vertex v in G_1.*

Then there exists a 1-factorization of G.

Lemma 3. *Let $a = 4n + 2$. Let $1 \leqslant r < a$. There exists a C_4-factorization of $G = rK_a \vee rK_a - E(F)$, where F is a 1-factor of G when r is odd and F has no edges when r is even.*

Proof. Let the vertex set of $rK_a \vee rK_a$ be $\mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let the edges joining vertices with different third coordinates be known as *mixed* edges, and all other edges be known as *pure* edges. Let $r = 4s + \varepsilon$ where $\varepsilon \in \{0, 1, 2, 3\}$.

Let $\{H_1, \ldots, H_n\}$ be a Hamilton decomposition of K_{2n+1} on the vertex set $\mathbb{Z}_{2n+1}$. For $1 \leqslant j \leqslant n - s - \lceil(\varepsilon-2)/4\rceil = \alpha$, if $H_j = (v_0, v_1, \ldots, v_{2n})$ then define

$$C_{2j-1} = \{((v_\ell, 0, 0), (v_{\ell+1}, 0, 1), (v_\ell, 1, 0), (v_{\ell+1}, 1, 1)) \mid 0 \leqslant \ell \leqslant 2n\}$$

and

$$C_{2j} = \{((v_{\ell+1}, 0, 0), (v_\ell, 0, 1), (v_{\ell+1}, 1, 0), (v_\ell, 1, 1)) \mid 0 \leqslant \ell \leqslant 2n\}.$$

Then each of $C_1, C_2, \ldots, C_{2\alpha}$ is a C_4-factor of $K_a \vee K_a$ which contains only mixed edges; in fact, the mixed edges in $C_{2j-1} \cup C_{2j}$ are precisely the mixed edges corresponding to the edges in $\{\{(u, 0), (v, 0)\}, \{(u, 0), (v, 1)\}, \{(u, 1), (v, 0)\}, \{(u, 1), (v, 1)\} \mid \{u, v\} \in E(H_j)\}$.

Now let G be the graph with vertex set $\mathbb{Z}_{2n+1} \times \mathbb{Z}_2$ and edge set

$$E(G) = \{\{(u, 0), (v, 0)\}, \{(u, 0), (v, 1)\}, \{(u, 1), (v, 0)\}, \{(u, 1), (v, 1)\} \mid \{u, v\} \in E(H_j), \alpha + 1 \leqslant j < n\} \cup \{\{(u, 0), (u, 1)\} \mid u \in \mathbb{Z}_{2n+1}\}.$$

Then, by the definition of α, G is a β-regular graph, where

$$\beta = \begin{cases} 4s + 1 & \text{if } \varepsilon \leqslant 2 \text{ and} \\ 4s + 5 & \text{if } \varepsilon = 3. \end{cases}$$

Furthermore, the function $f((u, 0)) = (u, 1)$ is clearly an isomorphism from $G[\mathbb{Z}_{2n+1} \times \{0\}]$ to $G[\mathbb{Z}_{2n+1} \times \{1\}]$. Therefore, by Lemma 2, there exists a 1-factorization $\{F_1, \ldots, F_\beta\}$ of G. For $1 \leqslant i \leqslant \min\{r, \beta\}$ let S_i be a new C_4-factorization of K_{4n+2} on the vertex set $\mathbb{Z}_{2n+1} \times \mathbb{Z}_2$ in which $F(S_i) = F_i$. Now use this to form $\mathscr{F}(S_i)$, a C_4-factorization of the graph formed by joining two copies of K_{4n+2} (on the vertex set $\mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \times \mathbb{Z}_2$) by the 1-factor consisting of the mixed edges corresponding to F_i. Let $B = (\bigcup_{1 \leqslant j \leqslant 2\alpha} C_j) \cup (\bigcup_{1 \leqslant i \leqslant \min\{r,\beta\}} \mathscr{F}(S_i))$.

The only case where $\beta < r$ is when $r = 4s + 2$, so $\varepsilon = 2$ and so $\beta = 4s + 1$. So cycles in B cover each pure edge exactly r times except when $\varepsilon = 2$; in this exceptional case each pure edge is covered exactly $r - 1$ times. All mixed edges are covered by cycles in B exactly once except for the mixed edges corresponding to edges in $\{F_{\min\{f,\beta\}+1}, \ldots, F_\beta\}$ and except for the horizontal mixed edges.

If $\varepsilon \neq 2$, now consider the use of the mixed edges corresponding to the edges in $F_{r+1}, \ldots, F_\beta$. If $\varepsilon = 0$ then let D be the C_4-factor formed by the corresponding mixed edges to those in $F_\beta = F_{r+1}$ together with the horizontal mixed edges. Then $B \cup D$ is a C_4-factorization of $rK_a \vee rK_a$. If $\varepsilon = 1$ then B is a C_4-factorization of the graph formed from $rK_a \vee rK_a$ by removing the edges in the 1-factor consisting of the horizontal edges. If $\varepsilon = 3$ then let D be the C_4-factor formed by the corresponding mixed edges to those in F_β together with the horizontal mixed edges. Then $B \cup D$ is a C_4-factorization of the graph formed from $rK_a \vee rK_a$ by removing edges in the 1-factor consisting of the corresponding mixed edges to those in $F_{\beta-1}$.

Finally, consider the case where $\varepsilon = 2$. As stated above, all pure edges still need to be used once, as do the horizontal mixed edges. But, as in the proof of Lemma 1, $\mathscr{F}'(S_\beta)$ covers precisely these edges. So $B \cup \mathscr{F}'(S_\beta)$ is a C_4-factorization of $rK_a \vee rK_a$ $\square$

There is one glaring omission in what we have proved so far, namely the case in Lemma 3 where $r = 0$, so there are no pure edges; but that is easily handled by the following lemma.

Lemma 4. *Let a be even. There exists a C_4-factorization of the complete bipartite graph $K_{a,a}$.*

Proof. Let V be partitioned into parts in $\mathbb{Z}_a \times \mathbb{Z}_2$. Define the parallel class $C_i = \{((2j, 0), (2j + 2i, 1), (2j + 1, 0), (2j + 2i + 1, 1)) | j \in \mathbb{Z}_{a/2}\}$ for each $i \in \mathbb{Z}_{a/2}$. Then $\{C_1, \ldots, C_{a/2}\}$ is the required decomposition. $\square$

Finally, we easily handle the case where there are no mixed edges.

Lemma 5. *Let $a = 4n$. There exists a C_4-factorization of $\lambda_1 K_a - E(F)$, where F is a 1-factor when λ_1 is odd and F has no edges when λ_1 is even.*

Proof. Let K_a have vertex set $\mathbb{Z}_{2n} \times \mathbb{Z}_2$. Let $\{F_0, \ldots, F_{2n-2}\}$ be a 1-factorization of K_{2n} on the vertex set $\mathbb{Z}_{2n}$. Define $C(i) = \{((u, 0), (v, 1), (u, 1), (v, 0)) | \{u, v\} \in F_i\}$. Then $C(i)$ is a parallel class in K_a, and $C = \bigcup_{i \in \mathbb{Z}_{2n-1}} C(i)$ is a C_4-factorization of $K_a - F$.

Define $C'(0) = \{((u, 0), (u, 1), (v, 1), (v, 0)), ((u, 0), (v, 1), (v, 0), (u, 1)) | \{u, v\} \in F_0\}$. Then

$$\lambda_1 \bigcup_{i=1}^{2n-2} C(i) \cup \lceil \lambda_1/2 \rceil C(0) \cup \lfloor \lambda_1/2 \rfloor C'(0)$$

is a C_4-factorization of $\lambda_1 K_a - F$, where F is the 1-factor $\{\{(u, 0), (u, 1)\} | u \in \mathbb{Z}_{2n}\}$ if λ_1 is odd, and F is empty if λ_1 is even. $\square$

3. The main result

Theorem 2. *Let $\lambda_1, \lambda_2 \geqslant 1$, a be even and $p \geqslant 2$. Let $G = K(a, p; \lambda_1, \lambda_2)$.*
There exists a C_4-factorization of G (of $G - F$, where F is a 1-factor of G) if and only if

(1) *4 divides ap*;
(2) *λ_1 is even (for G); λ_1 is odd (for $G - F$)*;
(3) *if $a \equiv 2 \pmod 4$ then $\lambda_2 a(p - 1) \geqslant \lambda_1$, unless $a = 2$ and λ_1 is odd, in which case $\lambda_2 a(p - 1) \geqslant \lambda_1 - 1$.*

Proof. Let the vertex set V of G be partitioned into parts $V_1, \ldots, V_p$, each of size a. We begin by proving the necessity of conditions (1)–(3).

Condition (1) follows because each parallel class naturally induces a partition of V into sets of size 4. Condition (2) holds because each vertex has degree $\lambda_1(a-1) + \lambda_2 a(p-1)$ in G, which must clearly be even for a C_4-factorization of G, and odd for a C_4-factorization of $G - F$. Notice that conditions (1)–(2) imply that $\lambda_1 pa(a-1)/2 + \lambda_2 a^2 p(p-1)/2$, the number of edges in G, is divisible by 4. Similarly, conditions (1)–(2) imply that the number of edges in $G - F$, namely, $(\lambda_1(a - 1) - 1)/(ap/2) + \lambda_2 a^2 p(p - 1)/2$, is also divisible by 4.

To see that condition (3) is necessary, note that when $a \equiv 2 \pmod 4$, p must be even so that the number of vertices in each parallel class is divisible by 4. Also, each parallel class P *must* contain at least *two* vertices in each part that are incident with a mixed edge in P, so P must contain at least p mixed edges. Since each parallel class contains ap edges altogether, the number of parallel classes in G is $\lambda_1(a - 1)/2 + \lambda_2 a(p - 1)/2$, the total number of edges divided by ap. Similarly, the number of parallel classes in $G - F$ is $(\lambda_1(a - 1) - 1)/2 + \lambda_2 a(p - 1)/2$. So the number of mixed edges in G satisfies

$$\lambda_2 a^2 p(p - 1)/2 \geqslant \lambda_1(a - 1)p/2 + \lambda_2 ap(p - 1)/2 \quad \text{so}$$

$$\lambda_2 a(a - 1)(p - 1) \geqslant \lambda_1(a - 1) \quad \text{and so}$$

$$\lambda_2 a(p - 1) \geqslant \lambda_1.$$

Similarly, the number of mixed edges in $G - F$ satisfies $\lambda_2 a(p-1) \geqslant \lambda_1 - 1/(a-1)$. Since the left-hand side of this inequality is an integer, this inequality is the same as the one for G unless $a = 2$, so condition (3) is necessary.

To prove the sufficiency of conditions (1)–(3), we begin by assuming that $a \equiv 0 \pmod 4$. Let $a = 4n$. Then by Lemma 5 there exists a C_4-factorization $C(i)$ of $\lambda_1 K_a$ (if λ_1 is even) or of $\lambda_1 K_a - E(F)$ where F is a 1-factor of K_a (if λ_1 is odd) on the vertex set V_i for $a \leqslant i \leqslant p$. So clearly $\bigcup_{1 \leqslant i \leqslant a} C_i$ can be used to form a set C of parallel classes of G (or $G - F$) which uses each pure edge λ_1 times.

By [7] there exists a 1-factorization $\{F_1, \dots, F_z\}$ of the complete multipartite graph with p parts, each of size $2n$ (so $z = 2n(p-1)$). Then $C(F_k) = \{(u, v, u + 2n, v + 2n) | \{u, v\} \in F_k\}$ is a C_4-factor of G that contains only mixed edges. Furthermore, the multiset C' consisting of λ_2 copies of $\bigcup_{1 \leqslant k \leqslant z} C(F_k)$ is a C_4-factorization of the complete multipartite graph with p parts and $4n$ vertices in each part. Then $C \cup C'$ is a C_4-factorization of G (or of $G - F$).

Next suppose that $a \equiv 2 \pmod 4$. Let $a = 4n + 2$. Then by (1), p is even, so let $\{F_1, \dots, F_{p-1}\}$ be a 1-factorization of K_p on the vertex set $\{1, \dots, p\}$. If λ_1 is even, by (3), there exist *even* integers $l_{i,j}$ for $1 \leqslant i \leqslant \lambda_2$ and $1 \leqslant j \leqslant p - 1$ such that:

(a) $0 \leqslant l_{i,j} \leqslant a$ for all i, j, and
(b) $\Sigma_{i=1}^{\lambda_2} \Sigma_{j=1}^{p-1} l_{i,j} = \lambda_1$.

And if λ_1 is odd, other than in the special case where $a = 2$ and $\lambda_2 a(p-1) = \lambda_1 - 1$, then by (3) there exist integers $l_{i,j}$ for $1 \leqslant i \leqslant \lambda_2$ and $1 \leqslant j \leqslant p - 1$ with all but one of the integers $l_{i,j}$ even, and one odd, so that (a) and (b) above also hold. (Of course in this case, at least one integer $l_{i,j}$ is strictly less than a, since λ_1 is odd and (3) holds.) If we are in the special case where $a = 2$ and $\lambda_2 a(p-1) = \lambda_1 - 1$, then we can instead find a C_4-factorization of $G' = K(a, p; \lambda_1 - 1, \lambda_2)$ (for which (3) is clearly satisfied) as described next, then set F to be the pure edges in G that are not in G'.

For $1 \leqslant i \leqslant \lambda_2$ and $1 \leqslant j \leqslant p - 1$, and for each $e = \{u, v\} \in F_j$, define a set $C_{i,j}(e)$ of partial parallel classes of G formed by applying Lemma 1 if $l_{i,j} = a$, Lemma 3 if $1 \leqslant l_{i,j} < a$, and Lemma 4 if $l_{i,j} = 0$, where the vertex sets in the two copies of K_a are V_u and V_v. Then $\bigcup_{e \in F_j} C_{i,j}(e)$ is a C_4-factorization of the subgraph of G formed by joining each pair of vertices in the same part of G with $l_{i,j}$ edges, and joining each pair of vertices in parts V_u and V_v with one edge, for each $e = \{u, v\} \in F_j$. Therefore

$$\bigcup_{1 \leqslant j \leqslant p-1} \left(\bigcup_{1 \leqslant i \leqslant \lambda_2} \cup_{e \in F_j} C_{i,j}(e) \right)$$

is the required C_4-factorization of G, or of $G - F$ when λ_1 is odd. $\square$

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